Birational rigidity of complete intersections

Fumiaki Suzuki
Graduate School of Mathematical Sciences, The University of Tokyo

Background

Recall the following definition :

Definition

A Mori fiber space X/S is called *birationally superrigid* if any birational map to the source of another Mori fiber space is isomorphism.

It implies that X is **non-rational** and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

Consider a complete intersection X of type $X_{d_1,\cdots,d_s} \subset \mathbb{P}^{\Sigma_{i=1}^s d_i}$ of dimesion ≥ 3 with only mild singularities, which is defined by s hypersurfaces of degree $d_1, \cdots, d_s$ in a projective space $\mathbb{P}^{\Sigma_{i=1}^s d_i}$. It is Fano of index 1 and rationally-connected.
For $s = 1$, after the works of Iskovskih-Manin, Pukhlikov and Chel'tsov, de Fernex proved :

Theorem (de Fernex '13)

For $N \geq 4$, every smooth hypersurface $X = X_N \subset \mathbb{P}^N$ of degree N is birationally superrigid.

For $s \geq 2$, its birational superrigidity is known only when X is one of the following:

Known cases when $s \geq 2$

- a smooth complete intersection $X = X_{d_1,\cdots,d_s} \subset \mathbb{P}^{\Sigma_{i=1}^s d_i}$ of dimension ≥ 12 which satisfies so-called regularity conditions, except three infinite series $X_{2,\cdots,2}$, $X_{2,\cdots,2,3}$ and $X_{2,\cdots,2,4}$, by Pukhlikov,
- a smooth complete intersection $X = X_{2,4} \subset \mathbb{P}^6$ not containing planes by Chel'tsov.

No explicit examples which satisfy these conditions have been obtained so far.

Birational superrigid complete intersections

We prove birational superrigidity of smooth and singular complete intersections. For s positive integers $d_1, \cdots, d_s$, set

$$c_s(d_1, \cdots, d_s) = \frac{2(\Sigma_{i=1}^s d_i + 1)}{\sqrt{\Pi_{i=1}^s d_i}} - 5s$$

in what follows.

Smooth complete intersections

Theorem A

Let $d_1, \cdots, d_s \geq 2$ be integers which satisfy

$$1 \leq c_s(d_1, \cdots, d_s).$$

Then every smooth complete intersection $X = X_{d_1,\cdots,d_s} \subset \mathbb{P}^{\Sigma_{i=1}^s d_i}$ is birationally superrigid.

We obtain explicit examples of birationally superrigid complete intersections which are not hypersurfaces. Here we only give the simplest ones among them.

Corollary

Every smooth complete intersection

$$X = X_{2,d} \subset \mathbb{P}^{d+2}, X_{3,d} \subset \mathbb{P}^{d+3}, X_{4,d} \subset \mathbb{P}^{d+4}, X_{2,2,d} \subset \mathbb{P}^{d+4}$$

is birationally superrigid for $d \geq 55, 83, 111, 246$ respectively.

Singular complete intersections : Case 1

Recall that an isolated singularity is called *a semi-homogeneous hypersurface singularity* if its tangent cone is a hypersurface which is smooth away from the vertex.

Theorem B

For positive integers $d_1, \cdots, d_s \geq 2$, every complete intersection $X = X_{d_1,\cdots,d_s} \subset \mathbb{P}^{\Sigma_{i=1}^s d_i}$ with only semi-homogeneous hypersurface singularities of multiplicity at most $c_s(d_1, \cdots, d_s) - 2$ is birationally superrigid.

Singular complete intersections : Case 2

For a complete intersection $X \subset \mathbb{P}^N$, denote by $X^\vee \subset (\mathbb{P}^N)^*$ the dual variety of X. Set $d = \deg X$ and $\pi = \Sigma_{i_1+\cdots+i_s=\dim X}(d_{i_1}-1)^{i_1}\cdots(d_{i_s}-1)^{i_s}$. It is known that $X^\vee$ is a hypersurface and $\deg X^\vee = d\pi$ if X is smooth.

Theorem C

Let $d_1, \cdots, d_s \geq 2$ be positive integers and $X = X_{d_1,\cdots,d_s} \subset \mathbb{P}^{\Sigma_{i=1}^s d_i}$ be a singular complete intersection with t isolated hypersurface singularities. If $X^\vee$ is a hypersurface and

$$d\pi - \deg X^\vee \leq c_s(d_1, \cdots, d_s) + 2t - 5,$$

then X is birationally superrigid.

Pukhlikov's multiplicity bounds

The following is a key proposition, which was known when $s = 1$ by Pukhlikov and $k = 1$ by Chel'tsov.

Proposition

Let X be a complete intersection in $\mathbb{P}^N$ defined by s hypersurfaces and α be an effective cycle on X of pure codimension k such that $\alpha \sim m \cdot c_1(\mathcal{O}_X(1))^k \cap [X]$. Assume either that X is smooth or $ks + \dim \mathrm{Sing}(X) + 1 < N$. Then $e_S(\alpha) \leq m$ for every closed subvariety $S \subset X$ of dimension ks not meeting the singular locus of X.

Proof: We may assume that $\dim S = ks$ and $S \in |\alpha|$. Then we use the method of multiple residual intersection, to construct a cycle $\mathbb{R}$ of pure-dimensional k on X such that

(1) $|\mathbb{R}| \subset X^{sm}$,
(2) α and $\mathbb{R}$ intersect properly on X, i.e. $\dim |\alpha| \cap |\mathbb{R}| = 0$,
(3) $S \cap |\mathbb{R}|$ contains at least $\deg \mathbb{R}$ points.

Then

$$\begin{aligned} m \deg \mathbb{R} &= \alpha \cdot \mathbb{R} \\ &\geq \sum_{t \in S \cap |\mathbb{R}|} i(t, \alpha \cdot \mathbb{R}; X) \\ &\geq \sum_{t \in S \cap |\mathbb{R}|} e_t(\alpha) \cdot e_t(\mathbb{R}) \\ &\geq e_S(\alpha) \cdot \deg \mathbb{R} \end{aligned}$$

The proof is done.